Siberian Mathematical Journal, Vol. 46, *No.* 6, *pp.* 1039–1049, 2005
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A BRANCHING METHOD FOR STUDYING STABILITY OF A SOLUTION TO A DELAY DIFFERENTIAL EQUATION

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Abstract: We study stability of antisymmetric periodic solutions to delay differential equations. We introduce a one-parameter family of periodic solutions to a special system of ordinary differential equations with a variable period. Conditions for stability of an antisymmetric periodic solution to a delay differential equation are stated in terms of this period function.

Keywords: delay differential equation, periodic solution, stability

1. Statement of the Problem

Consider a scalar differential equation with constant delay

$$\frac{dx(t)}{dt} = -f(x(t-\tau)), \tag{1.1}$$

where f is a continuously differentiable odd function with positive derivative on an interval $(-\gamma, \gamma)$, $\gamma > 0$. In this article we study the questions of existence and stability for the periodic solutions satisfying the antisymmetry condition $x(t + 2\tau) = -x(t)$, $t \in (-\infty, +\infty)$. The isolated periodic solutions to delay differential equations were studied in [1, 2]; the periodic solutions with a period divisible by the delay, in [3, 4]; and their stability, in [4, 5].

Using the results of [5], we reduce the problem of finding an antisymmetric periodic solution to the delay differential equation (1.1) to solving the following special boundary value problem for a system of ordinary differential equations:

$$\dot{x_1} = f(x_2), \quad \dot{x_2} = -f(x_1), \tag{1.2}$$

$$x_1(\tau) = x_2(0), \quad x_2(\tau) = -x_1(0). \tag{1.3}$$

The connection between an antisymmetric periodic solution x to the delay differential equation (1.1) and a solution $\{x_1, x_2\}$ to the boundary value problem (1.2), (1.3) is now given by the formulas

$$x(t) = \begin{cases} x_1(t), & t \in [0, \tau], \\ x_2(t-\tau), & t \in [\tau, 2\tau]. \end{cases} \tag{1.4}$$

System (1.2) extends from the interval $[0, \tau]$ to the whole real axis and has the first integral

$$F(x_1) + F(x_2) = C = \text{const}, \quad x_1, x_2 \in (-\gamma, \gamma), \tag{1.5}$$

where $F(x) = \int_0^x f(z)\,dz$, $x \in (-\gamma, \gamma)$.

Formula (1.5) defines the position of the integral curves of system (1.2) on the phase plane. The special initial conditions $x_1(0, \mu) = 0$ and $x_2(0, \mu) = \mu$ are associated with the closed integral curves generating periodic motions $\{x_1(t, \mu), x_2(t, \mu)\}$, $t \in \mathbb{R}$, for $0 < \mu < \gamma$. The periods T of solutions to (1.2) depend on μ.

For small values of μ, (1.2) is a Lyapunov system [6, p. 153]. To find the asymptotic expansions of periodic solutions to this system for small values of μ, we may use the Lyapunov method [6, p. 159].

The authors were supported by the RAS Program “Mathematical Methods in Nonlinear Dynamics” (Grant 15).

Ekaterinburg. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 46, No. 6, pp. 1288–1301, November–December, 2005. Original article submitted June 25, 2003. Revision submitted April 28, 2005.

Assertion 1.1. *Let f be a thrice continuously differentiable odd function in a neighborhood of the point $x = 0$ such that $f'(0) > 0$. For small values of μ, the periodic solutions $\{x_1(t,\mu), x_2(t,\mu)\}$, $t \in \mathbb{R}$, with period T are defined by the formulas*

$$x_i(t,\mu) = y_i\left(\frac{2\pi}{T(\mu)}t, \mu\right), \quad t \in \mathbb{R},\ i = 1, 2,$$

where $y_i(s,\mu)$, $i = 1,2$, $s \in \mathbb{R}$, are 2π-periodic functions. For small values of μ the functions $y_1(s,\mu)$ and $y_2(s,\mu)$ on the interval $[0, 2\pi]$ and the period function T have the asymptotic expansions:

$$T(\mu) = \frac{2\pi}{f'(0)} - \frac{\pi f'''(0)}{4(f'(0))^2}\mu^2 + o(\mu^2),$$

$$y_1(s,\mu) = \sin(s)\mu + \frac{f'''(0)}{24 f'(0)}(-\sin^3(s)\cos^2(s) - \sin^5(s) + \sin(s))\mu^3 + o(\mu^3),$$

$$y_2(s,\mu) = \cos(s)\mu + \frac{f'''(0)}{24 f'(0)}(-\sin^2(s)\cos^3(s) - \cos^5(s) + \cos(s))\mu^3 + o(\mu^3).$$

Assertion 1.2. *Let f be a continuously differentiable odd function with positive derivative on an interval $(-\gamma, \gamma)$. For the delay differential equation (1.1) to have a periodic solution satisfying the antisymmetry condition, it is necessary and sufficient that the number 4τ belong to the interval $(\inf_{0<\mu<\gamma} T(\mu), \sup_{0<\mu<\gamma} T(\mu))$. This interval is complemented with the point $t = \inf_{0<\mu<\gamma} T(\mu)$ ($t = \sup_{0<\mu<\gamma} T(\mu)$) if $\inf_{0<\mu<\gamma} T(\mu) = T(\mu_1)$ ($\sup_{0<\mu<\gamma} T(\mu) = T(\mu_2)$) for some $\mu_1 \in (0,\gamma)$ ($\mu_2 \in (0,\gamma)$).*

Under the conditions of Assertion 1.2, the antisymmetric periodic solution $x(t,\mu^*)$, $t \in \mathbb{R}$, to the delay differential equation is expressed by (1.4) in terms of the periodic solution $\{x_1(t,\mu^*),\ x_2(t,\mu^*)\}$, $t \in \mathbb{R}$, to the system (1.2) of ordinary differential equations for which $T(\mu^*) = 4\tau$, $0 < \mu^* < \gamma$. The delay differential equation (1.1) may have several such solutions. Their number is determined by the number of positive solutions to the equation $T(\mu) = 4\tau$ for $0 < \mu < \gamma$. The delay differential equation has a unique periodic solution if the period function T is monotone.

To study stability of an antisymmetric periodic solution $x(t,\mu^*)$, $t \in \mathbb{R}$, to the delay differential equation (1.1), consider the linear approximation equation

$$\frac{dy(t)}{dt} = -f'(x(t - T(\mu^*)/4, \mu^*))y(t - T(\mu^*)/4) \tag{1.6}$$

for the equation of perturbed motion. In (1.6), the function $a(t,\mu^*) = f'(x(t - T(\mu^*)/4, \mu^*))$, $t \in \mathbb{R}$, takes only positive values and depends periodically on t with period $T(\mu^*)/2$.

In this article we propose to use the branching approach to the problem of studying stability of a periodic solution. This method is connected with passage from studying stability of the delay differential equation (1.6) to studying stability of the one-parameter family of delay differential equations

$$\frac{dy(t)}{dt} = -f'(x(t - T(\mu)/4, \mu))y(t - T(\mu)/4), \quad \mu \in [0,\gamma). \tag{1.7}$$

2. Stability of a Linear Periodic System of Delay Differential Equations

Changing variables in the delay differential equation (1.7):

$$t = \frac{T(\mu)}{2\pi}s, \quad \tilde{y}(s) = y\left(\frac{T(\mu)}{2\pi}s\right), \quad \tilde{x}(s,\mu) = x\left(\frac{T(\mu)}{2\pi}t, \mu\right),$$

we find that

$$\frac{d\tilde{y}(s)}{ds} = -\frac{T(\mu)}{2\pi}f'(\tilde{x}(s - \pi/2, \mu))\tilde{y}(s - \pi/2), \quad \mu \in [0,\gamma). \tag{2.1}$$

The coefficient of the so-obtained differential equation is close to a constant for small μ; i.e., the delay differential equation is quasiharmonic.

Assertion 2.1. *Suppose that the conditions of Assertion* 1.1 *are satisfied. Then, for small positive values of* μ*, the quasiharmonic delay differential equation* (2.1) *is stable if* $T''(0) > 0$ *and unstable if* $T''(0) < 0$.

PROOF. For $\mu = 0$, the delay differential equation (2.1) has the form $\frac{d\hat{y}(s)}{ds} = -\hat{y}(s - \pi/2)$. Write down the characteristic equation of (2.1): $\lambda = -e^{-(\pi/2)\lambda}$. Using the D-partition method, we can show that it has only two purely imaginary roots $\lambda = \pm i$, whereas the other roots have negative real parts. Consequently, stability of (2.1) for small positive μ is determined by the behavior of the characteristic exponents of this equation which coincide with the numbers $\pm i$ for $\mu = 0$. To the pair of purely imaginary roots of the characteristic equation, there corresponds one semisimple characteristic exponent $\lambda_0 = i$ of (2.1). It is associated with two linearly independent Floquet solutions: $y_1(s) = e^{is}$ and $y_2(s) = e^{-is}$, $s \in \mathbb{R}$. The quasiharmonic delay differential equation (2.1) has a π-periodic solution $y(s, \mu) = \dot{x}(s, \mu)$, $s \in \mathbb{R}$, which is a Floquet solution to this equation with the characteristic exponent $\lambda = i$. Since the characteristic exponent λ_0 is double, the quasiharmonic delay differential equation (2.1) has one more characteristic exponent $\lambda(\mu)$ ($\lambda(0) = \lambda_0 = i$). If the real part of this characteristic exponent is greater than zero for small positive values of μ then the quasiharmonic equation is unstable. If the real part of the characteristic exponent $\lambda(\mu)$ is less than zero for small positive values of the parameter μ then the critical characteristic exponent $\lambda = i$ is simple, the other characteristic exponents have negative real parts, and the quasiharmonic delay differential equation is stable. Using Assertion 1.1 and the method for calculation of the characteristic exponents for quasiharmonic delay differential equations [7], we find that

$$\lambda(\mu) = i - \frac{f'(0)T''(0)}{4 + \pi^2}\mu^2 + o(\mu^2).$$

The assertion is proven.

Stability of (2.1) at the large values of μ depends on the disposition of the eigenvalues of the monodromy operator acting in $C[-\pi, 0]$. The monodromy operator is defined by the formula $(U\varphi)(\vartheta) = y(\pi + \vartheta, \varphi)$, $\vartheta \in [-\pi, 0]$, where $y(\pi + \cdot, \varphi)$ is a segment of the solution to (2.1) with the initial moment $t = 0$ and the initial function φ.

We can replace the problem of finding nonzero eigenvalues $\rho \in \mathbb{C}$ of the monodromy operator U with the problem of finding nonzero eigenvalues $z \in \mathbb{C}$ to the special boundary value problem [8, p. 49]

$$\dot{y}_1 = za(\vartheta, \mu)y_2, \quad \dot{y}_2 = -za(\pi/2 + \vartheta, \mu)y_1, \tag{2.2}$$

$$y_1(-\pi/2) = -zy_2(0), \quad y_2(-\pi/2) = zy_1(0), \tag{2.3}$$

where $\rho = -z^2$, $a(\vartheta, \mu) = \frac{T(\mu)}{2\pi} f'(\tilde{x}(\vartheta - \pi/2, \mu))$, $\vartheta \in [-\pi/2, \pi/2]$, $\mu \in [0, \gamma)$, $z, \rho \in \mathbb{C}$. Reduce the boundary value problem (2.2), (2.3) to the special form

$$J\dot{y} = zH(\vartheta, \mu)y, \tag{2.4}$$

$$y(-\pi/2) = zJy(0). \tag{2.5}$$

Here $y = (y_1, y_2)^\top$, $z \in \mathbb{C}$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H(\vartheta, \mu) = \begin{pmatrix} a(\pi/2 + \vartheta, \mu) & 0 \\ 0 & a(\vartheta, \mu) \end{pmatrix},$$

$\mu \in [0, \gamma)$ and $\vartheta \in [-\pi/2, 0]$. The eigenvalues z of the boundary value problem (2.4), (2.5) are determined from the characteristic equation

$$\det(zJY(0, z, \mu) - I_2) = 0, \tag{2.6}$$

where Y is the normed fundamental matrix of system (2.4), $Y(-\pi/2, z, \mu) = I_2$, $z \in \mathbb{C}$, $\mu \in [0, \gamma)$, I_2 is the identity matrix of order 2. Using Liouville's formula, we find that

$$\det(Y(\vartheta, z, \mu)) = \exp\biggl(\int_{-\pi/2}^{\vartheta} Tr(H(s, \mu))\, ds\biggr) = 1, \quad z \in \mathbb{C},\ \mu \in [0, \gamma),\ \vartheta \in [-\pi/2, 0].$$

Eventually, the characteristic equation (2.6) takes the form

$$D(z,\mu) = z^2 - 2V(z,\mu)z + 1 = 0, \tag{2.7}$$

where $V(z,\mu) = \frac{1}{2}(y_{12}(0,z,\mu) - y_{21}(0,z,\mu))$ and $Y(\vartheta,z,\mu) = \|y_{ij}(\vartheta,z,\mu)\|_1^2$, $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$. Consider some properties of the normed fundamental matrix Y of (2.4).

Lemma 2.1. *The following equality is valid for the fundamental matrix Y:*

$$Y(-\pi/2 - \vartheta, \mu, z) = JY(\vartheta,\mu,-z)J^\top Y(0,\mu,z), \tag{2.8}$$

where $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, and $\mu \in [0,\gamma)$.

Proof. Replacing the argument ϑ with ϑ_1, rewrite (2.2) as follows:

$$\dot y_1 = za(\vartheta_1,\mu)y_2, \quad \dot y_2 = -za(\pi/2 + \vartheta_1, \mu)y_1, \quad \vartheta_1 \in [-\pi/2, 0].$$

By the change of variables $\vartheta_1 = -\pi/2 - \vartheta$, $v_1(\vartheta) = y_2(-\pi/2 - \vartheta)$, $v_2(\vartheta) = -y_1(-\pi/2 - \vartheta)$, $\vartheta \in [-\pi/2, 0]$, in this system of differential equations, we obtain

$$\dot v_1 = -za(\vartheta,\mu)v_2, \quad \dot v_2 = za(\pi/2 + \vartheta, \mu)v_1. \tag{2.9}$$

An arbitrary solution $\{v_1, v_2\}^\top$ to the system (2.9) of differential equations is representable as

$$(v_1(\vartheta,\mu,z), v_2(\vartheta,\mu,z))^\top = J^\top y(-\pi/2 - \vartheta, \mu, z), \tag{2.10}$$

where $y(\vartheta,\mu,z) = (y_1(\vartheta,\mu,z), y_2(\vartheta,\mu,z))^\top$, $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, is some solution to (2.2). Then

$$(v_1(\vartheta,\mu,z), v_2(\vartheta,\mu,z))^\top = J^\top Y(-\pi/2 - \vartheta, \mu, z)D,$$

with $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, and $D \in \mathbb{C}^2$. In view of the arbitrariness of the solution $\{v_1, v_2\}^\top$, the vector constant D can take arbitrary values in $\mathbb{C}^2$. The solution $\{v_1, v_2\}^\top$ also admits the representation

$$(v_1(\vartheta,\mu,z), v_2(\vartheta,\mu,z))^\top = y(\vartheta,\mu,-z) = Y(\vartheta,\mu,-z)C \tag{2.11}$$

for some $C \in \mathbb{C}^2$. The following is valid:

$$Y(\vartheta,\mu,-z)C = J^\top Y(-\pi/2 - \vartheta, \mu, z)D, \quad \vartheta \in [-\pi/2, 0],\ z \in \mathbb{C},\ \mu \in [0,\gamma).$$

Putting $\vartheta = -\pi/2$ in the above equality, we find that $C = J^\top Y(0,\mu,z)D$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$. Consequently,

$$Y(\vartheta,\mu,-z)J^\top Y(\vartheta,\mu,z)D = J^\top Y(-\pi/2 - \vartheta, \mu, z)D,\ \vartheta \in [-\pi/2, 0],\ z \in \mathbb{C},\ \mu \in [0,\gamma).$$

In view of the arbitrariness of $D \in \mathbb{C}^2$, the last equality implies (2.8).

Lemma 2.2. *The following is valid for the fundamental matrix Y:*

$$Y(-\pi/2 - \vartheta, \mu, z) = SY(\vartheta,\mu,z)Y(0,\mu,z), \tag{2.12}$$

where $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. By the change of variables $\vartheta_1 = -\pi/2 - \vartheta$, $v_1(\vartheta) = y_2(-\pi/2 - \vartheta)$, $v_2(\vartheta) = y_1(-\pi/2 - \vartheta)$, $\vartheta \in [-\pi/2, 0]$, in (2.2), we obtain the new system of differential equations

$$\dot v_1 = za(\vartheta,\mu)v_2, \quad \dot v_2 = -za(\pi/2 + \vartheta, \mu)v_1.$$

An arbitrary solution $\{v_1(\vartheta,\mu,z), v_2(\vartheta,\mu,z)\}^\top$, $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, to the new system of differential equations is representable as

$$(v_1(\vartheta,\mu,z), v_2(\vartheta,\mu,z))^\top = Y(\vartheta,\mu,z)C,$$

where $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, and $C \in \mathbb{C}^2$. But the same solution is also representable as

$$(\bar v_1(\vartheta,\mu,z), \bar v_2(\vartheta,\mu,z))^\top = SY(-\pi/2 - \vartheta, \mu, z)D,$$

where $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$, and $D \in \mathbb{C}^2$. Then

$$Y(\vartheta,\mu,z)C = SY(-\pi/2 - \vartheta, \mu, z)D$$

for all $\vartheta \in [-\pi/2, 0]$, $z \in \mathbb{C}$, $\mu \in [0,\gamma)$. It follows from this equality with $\vartheta = -\pi/2$ that $C = SY(0,\mu,z)D$. Then

$$Y(\vartheta,\mu,z)SY(0,\mu,z)D = SY(-\pi/2,\mu,z)D, \quad \vartheta \in [-\pi/2, 0],\ z \in \mathbb{C},\ \mu \in [0,\gamma).$$

Since $D \in \mathbb{C}^2$ is arbitrary, we arrive at (2.12).

Lemma 2.3. *Let f be a continuously differentiable odd function with positive derivative on an interval $(-\gamma,\gamma)$. Then the function V satisfies the conditions:*

(a) $V(1,\mu)=1$, $\mu\in[0,\gamma)$;

(b) $V(-z,\mu)=-V(z,\mu)$, $z\in\mathbb{C}$, $\mu\in[0,\gamma)$.

PROOF. We can show that (2.1) has an antisymmetric 2π-periodic solution $\tilde{y}(s,\mu)=\frac{d\tilde{x}(s,\mu)}{ds}$, $s\in\mathbb{R}$, $\mu\in[0,\gamma)$. It is associated with the eigenvalue $\rho=-1$ of the monodromy operator, since the identity $\dot{x}(\pi/2+\vartheta,\mu^*)\equiv-\dot{x}(\vartheta,\mu^*)$, $\vartheta\in[-\pi/2,0]$, holds for an antisymmetric periodic solution to the delay differential equation (1.1). To this number there correspond the two eigenvalues $z=\pm1$ of the boundary value problem (2.4), (2.5). It follows from (2.7) that V satisfies the conditions $V(\pm1,\mu)=\pm1$, $\mu\in[0,\gamma)$. The first assertion of the lemma is proven.

Show that $y_{12}(0,\mu,z)=-y_{21}(0,\mu,z)$, $\mu\in[0,\gamma)$, $z\in\mathbb{C}$. By Lemma 2.2, $Y(0,\mu,z)SY(0,\mu,z)=S$, $\mu\in[0,\gamma)$, $z\in\mathbb{C}$, or, in expanded form,

$$\begin{pmatrix} a_{11}(\mu,z) & a_{12}(\mu,z)\\ a_{21}(\mu,z) & a_{22}(\mu,z)\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

where

$$a_{11}(\mu,z)=y_{11}(0,\mu,z)(y_{21}(0,\mu,z)+y_{12}(0,\mu,z)),$$
$$a_{12}(\mu,z)=y_{12}^2(0,\mu,z)+y_{11}(0,\mu,z)y_{22}(0,\mu,z),$$
$$a_{21}(\mu,z)=y_{11}(0,\mu,z)y_{22}(0,\mu,z)+y_{21}^2(0,\mu,z),$$
$$a_{22}(\mu,z)=(y_{12}(0,\mu,z)+y_{21}(0,\mu,z))y_{22}(0,\mu,z),$$
$$\mu\in[0,\gamma),\quad z\in\mathbb{C}.$$

Let $y_{12}(0,\mu,z)+y_{21}(0,\mu,z))\neq0$ for some $\mu\in[0,\gamma)$, $z\in\mathbb{C}$. Then the matrix equality implies that

$$y_{11}(0,\mu,z)=y_{22}(0,\mu,z)=0,\quad y_{12}(0,\mu,z)=y_{21}(0,\mu,z)=\pm1.$$

In this case $\det Y(0,\mu,z)=-1$; a contradiction. Consequently,

$$Y(0,\mu,z)=\begin{pmatrix} y_{11}(0,\mu,z) & y_{12}(0,\mu,z)\\ -y_{12}(0,\mu,z) & y_{22}(0,\mu,z)\end{pmatrix},\quad \mu\in[0,\gamma),\ z\in\mathbb{C}.$$

Now, by Lemma 2.1, $Y(0,-z)=J^\top Y^{-1}(0,z)J$, or in expanded form,

$$\begin{pmatrix} y_{11}(0,\mu,-z) & y_{12}(0,\mu,-z)\\ -y_{12}(0,\mu,-z) & y_{22}(0,\mu,-z)\end{pmatrix}=\begin{pmatrix} y_{11}(0,\mu,z) & -y_{12}(0,\mu,z)\\ y_{12}(0,\mu,z) & y_{22}(0,\mu,z)\end{pmatrix},$$

where $\mu\in[0,\gamma)$ and $z\in\mathbb{C}$. From the matrix equality we find that

$$y_{12}(0,\mu,-z)=-y_{12}(0,\mu,z),\quad \mu\in[0,\gamma),\ z\in\mathbb{C}.$$

Consequently,

$$V(-z,\mu)=y_{12}(0,\mu,-z)=-y_{12}(0,\mu,z)=-V(z,\mu),\quad \mu\in[0,\gamma),\ z\in\mathbb{C}.$$

The lemma is proven.

As μ varies, the roots of (2.7) move over the complex plane in the symmetric manner with respect to the imaginary axis. Therefore, studying their motion, we can only consider the right half-plane. It follows from Lemma 2.3 that (2.7) has the roots $z=\pm1$; moreover, for each real root $z_1\neq1$, there is a real root $-z_1$ and, for each complex root z_2, there are complex roots $-z_2$, $\bar{z}_2$, and $-\bar{z}_2$.

Lemma 2.4. *Let f be a continuously differentiable odd function with positive derivative on an interval $(-\gamma, +\gamma)$. If a root z of (2.7) satisfies the condition $|z| = 1$ then either $z = 1$ or $z = -1$.*

Proof. Consider the following boundary value problem:

$$J\dot{y} = \lambda H(\vartheta, \mu)y, y(-\pi/2) = zJy(0), \quad |z| = 1, \ \lambda, z \in \mathbb{C}. \tag{2.13}$$

For fixed values of the arguments $\vartheta \in [-1, 0]$ and $\mu \in [0, \gamma)$ the matrices $H(\vartheta, \mu)$ are positive definite. Hence, the auxiliary boundary value problem (2.13) is selfadjoint. In view of this fact, (2.13) has at most countably many eigenvalues λ which may have only one limit point $\lambda = \infty$. The eigenvalues (if any) are real and have finite multiplicity [9, p. 176]. If the boundary value problem (2.4), (2.5) has an eigenvalue z satisfying the condition $|z| = 1$ then (2.13) has the eigenvalue $\lambda = z$. Since all eigenvalues of (2.13) are real, we conclude that the eigenvalue in question is either $z = 1$ or $z = -1$. The lemma is proven.

By Lemma 2.4, the roots of the characteristic equation, moving continuously on the complex plane, may meet the unit circle $|z| = 1$ only at the points $z = 1$ and $z = -1$. By symmetry of motion of the roots of the characteristic equation, the two roots intersect simultaneously the circle at the points $z = 1$ and $z = -1$. Also, the directions of intersection of the circle coincide. The root $z = 1$ becomes multiple at the intersection moment. This condition enables us to determine the corresponding value of the parameter $\mu = \mu^*$. We have

$$\frac{\partial D(1, \mu^*)}{\partial z} = 2 - 2V(1, \mu^*) - 2\frac{\partial V(1, \mu^*)}{\partial z} = -2\frac{\partial V(1, \mu^*)}{\partial z} = 0.$$

Consequently, the traversal of the roots of the characteristic equation on the complex plane through the unit circle occurs at the values of the parameter μ determined by the equation

$$\frac{\partial V(1, \mu)}{\partial z} = 0, \quad \mu \in [0, \gamma). \tag{2.14}$$

Consider a neighborhood of the point $z = 1$. Put $z = 1 + \tilde{z}$, where $\tilde{z}$ is a small perturbation. We search for the fundamental matrix of (2.4) in the form of an asymptotic expansion

$$Y(\vartheta, z, \mu) = Y_0(\vartheta, \mu) + Y_1(\vartheta, \mu)\tilde{z} + Y_2(\vartheta, \mu)\tilde{z}^2 + o(\tilde{z}^2),$$

where $\vartheta \in [-\pi/2, 0]$, $\tilde{z} \in \mathbb{C}$, and $\mu \in [0, \gamma)$. For finding the coefficients of this asymptotic expansion we have the equations

$$\dot{Y}_0 = J^{-1}H(\vartheta, \mu)Y_0, \tag{2.15}$$
$$\dot{Y}_1 = J^{-1}H(\vartheta, \mu)Y_1 + J^{-1}H(\vartheta, \mu)Y_0,$$
$$\dot{Y}_2 = J^{-1}H(\vartheta, \mu)Y_2 + J^{-1}H(\vartheta, \mu)Y_1, \quad \vartheta \in [-\pi/2, 0], \ \mu \in [0, \gamma).$$

Since $Y(-\pi/2, z, \mu) = I_2$, $z \in \mathbb{C}$, we have

$$Y_0(-\pi/2, \mu) = I_2, \quad Y_1(-\pi/2, \mu) = 0, \quad Y_2(-\pi/2, \mu) = 0, \quad \mu \in [0, \gamma).$$

Knowing Y_0, we can find the matrix functions

$$Y_1(\vartheta, \mu) = -\int_{-\pi/2}^{\vartheta} Y_0(\vartheta, \mu)Y_0^{-1}(s, \mu)JH(s, \mu)Y_0(s, \mu)\, ds, \tag{2.16}$$

$$Y_2(\vartheta, \mu) = -\int_{-\pi/2}^{\vartheta} Y_0(\vartheta, \mu)Y_0^{-1}(s, \mu)JH(s, \mu)Y_1(s, \mu)ds, \tag{2.17}$$

where $\vartheta \in [-\pi/2, 0)$ and $\mu \in [0, \gamma)$. The matrix Y_0 of the Hamilton equation satisfies the identity [9, p. 103]

$$Y_0^T(\vartheta, \mu)JY_0(\vartheta, \mu) \equiv J, \quad \vartheta \in [-\pi/2, 0], \ \mu \in [0, \gamma).$$

Therefore, the following formulas hold:

$$Y_1(\vartheta,\mu) = -Y_0(\vartheta,\mu) J \int\limits_{-\pi/2}^{\vartheta} Y_0^\top(s,\mu) H(s,\mu) Y_0(s,\mu)\, ds, \tag{2.18}$$

$$Y_2(\vartheta,\mu) = -Y_0(\vartheta,\mu) J \int\limits_{-\pi/2}^{\vartheta} Y_0^\top(s,\mu) H(s,\mu) Y_1(s,\mu)\, ds, \tag{2.19}$$

where $\vartheta \in [-\pi/2, 0]$ and $\mu \in [0,\gamma)$. The matrix function Y_0 is a fundamental matrix for the system of differential equations

$$\dot{\psi} = J^{-1} H(\vartheta,\mu)\psi,$$

where $\psi = (\psi_1,\psi_2)^\top$, $\vartheta \in [-\pi/2, 0]$, and $\mu \in [0,\gamma)$. Write down this system in the coordinate form

$$\dot{\psi}_1 = a(\vartheta,\mu)\psi_2, \quad \dot{\psi}_2 = -a(\pi/2+\vartheta,\mu)\psi_1, \quad \vartheta \in [-\pi/2, 0],\ \mu \in [0,\gamma). \tag{2.20}$$

Here

$$a(\vartheta,\mu) = \frac{T(\mu)}{2\pi} f'(\tilde{x}(\vartheta - \pi/2,\mu)) = \frac{T(\mu)}{2\pi} f'(\tilde{x}_1(\pi/2+\vartheta,\mu)),$$

$$a(\pi/2+\vartheta,\mu) = \frac{T(\mu)}{2\pi} f'(\tilde{x}(\vartheta,\mu)) = \frac{T(\mu)}{2\pi} f'(\tilde{x}_2(\pi/2+\vartheta,\mu)),$$

where $(\tilde{x}_1(s,\mu),\, \tilde{x}_2(s,\mu))^\top$, $s \in [0,\pi/2]$, $\mu \in [0,\gamma)$, is a solution to the system of differential equations

$$\frac{d\tilde{x}_1}{ds} = \frac{T(\mu)}{2\pi} f(\tilde{x}_2), \quad \frac{d\tilde{x}_2}{ds} = -\frac{T(\mu)}{2\pi} f(\tilde{x}_1).$$

Then (2.20) becomes

$$\dot{\psi}_1 = \frac{T(\mu)}{2\pi} f'(\tilde{x}_1(\pi/2+\vartheta,\mu))\psi_2, \quad \dot{\psi}_2 = -\frac{T(\mu)}{2\pi} f'(\tilde{x}_2(\pi/2+\vartheta,\mu))\psi_1. \tag{2.21}$$

Lemma 2.5. *Let f be a continuously differentiable odd function with positive derivative on an interval $(-\gamma,\gamma)$. Then the normed fundamental matrix of* (2.21) *has the form*

$$Y_0(\vartheta,\mu) = \left.\begin{pmatrix} \frac{\partial \tilde{x}_2(s,\mu)}{\partial \mu} + \frac{T'(\mu)}{2\pi}(s) f(\tilde{x}_1(s,\mu)) & \frac{f(\tilde{x}_1(s,\mu))}{f(\mu)} \\ -\frac{\partial \tilde{x}_1(s,\mu)}{\partial \mu} + \frac{T'(\mu)}{2\pi}(s) f(\tilde{x}_2(s,\mu)) & \frac{f(\tilde{x}_2(s,\mu))}{f(\mu)} \end{pmatrix}\right|_{s=\pi/2+\vartheta}, \quad \vartheta \in \mathbb{R},\ \mu \in (0,\gamma). \tag{2.22}$$

Proof. Consider the system of linear differential equations

$$\frac{dx_1}{dt} = f'(x_2(t,\mu))x_2, \quad \frac{dx_2}{dt} = -f'(x_1(t,\mu))x_1, \tag{2.23}$$

which is the system of differential equations in variations for (1.2). Using the methods of [5], we find its normed fundamental matrix:

$$X(t,\mu) = \begin{pmatrix} \frac{1}{f(\mu)} \frac{dx_1(t,\mu)}{dt} & \frac{\partial x_1(t,\mu)}{\partial \mu} \\ \frac{1}{f(\mu)} \frac{dx_2(t,\mu)}{dt} & \frac{\partial x_2(t,\mu)}{\partial \mu} \end{pmatrix}, \quad t \in \mathbb{R},\ \mu \in (0,\gamma).$$

By the change of variables

$$t = \frac{T(\mu)}{2\pi}s, \quad x_1\left(\frac{T(\mu)}{2\pi}s\right) = \tilde{x}_1(s), \quad x_2\left(\frac{T(\mu)}{2\pi}s\right) = \tilde{x}_2(s), \quad s \in \mathbb{R},\ \mu \in (0,\gamma),$$

from (2.23) we obtain

$$\frac{d\tilde{x}_1}{ds} = \frac{T(\mu)}{2\pi} f'(\tilde{x}_2(s,\mu))\tilde{x}_2, \quad \frac{d\tilde{x}_2}{ds} = -\frac{T(\mu)}{2\pi} f'(\tilde{x}_1(s,\mu))\tilde{x}_1, \tag{2.24}$$

where $s \in \mathbb{R}$ and $\mu \in (0,\gamma)$. Using the properties of the 2π-periodic solution

$$(\tilde{x}_1(s,\mu), \tilde{x}_2(s,\mu))^\top = \left(x_1\left(\frac{T(\mu)}{2\pi}s, \mu\right), x_2\left(\frac{T(\mu)}{2\pi}s, \mu\right)\right)^\top, \quad s \in \mathbb{R},\ \mu \in (0,\gamma),$$

to the system of differential equations

$$\frac{d\tilde{x}_1}{ds} = \frac{T(\mu)}{2\pi} f(\tilde{x}_2), \quad \frac{d\tilde{x}_2}{ds} = -\frac{T(\mu)}{2\pi} f(\tilde{x}_1), \quad s \in \mathbb{R},\ \mu \in (0,\gamma),$$

we find the fundamental matrix of (2.24):

$$\widetilde{X}(s,\mu) = \begin{pmatrix} \frac{f(\tilde{x}_2(s,\mu))}{f(\mu)} & \frac{\partial \tilde{x}_1(s,\mu)}{\partial \mu} - \frac{T'(\mu)}{2\pi} s f(\tilde{x}_2(s,\mu)) \\ -\frac{f(\tilde{x}_1(s,\mu))}{f(\mu)} & \frac{\partial \tilde{x}_2(s,\mu)}{\partial \mu} + \frac{T'(\mu)}{2\pi} s f(\tilde{x}_1(s,\mu)) \end{pmatrix},$$

$s \in \mathbb{R}$ and $\mu \in (0,\gamma)$. System (2.21) is adjoint to (2.24). Their solutions are connected by the transformation

$$(\tilde{x}_1(\pi/2+\vartheta,\mu), \quad \tilde{x}_2(\pi/2+\vartheta,\mu))^\top = J(\psi_1(\vartheta,\mu), \quad \psi_2(\vartheta,\mu))^\top, \ \vartheta \in \mathbb{R},\ \mu \in (0,\gamma).$$

We have the equality

$$JY_0(\vartheta,\mu)d = \widetilde{X}(\pi/2+\vartheta,\mu)c, \quad \vartheta \in \mathbb{R},\ \mu \in (0,\gamma),$$

where $c, d \in \mathbb{R}^2$. We now find $c = Jd$. Consequently, the fundamental matrix Y_0 has the form (2.22).

Lemma 2.6. *Let f be a continuously differentiable odd function with positive derivative on an interval $(-\gamma,\gamma)$. The root $z = 1$ of (2.7) is multiple if and only if $\mu = \mu^*$ is a critical point of the function T; i.e., $T'(\mu^*) = 0$. Moreover, the multiplicity of the root is two.*

Proof. Write down the expansion of the function

$$V(z,\mu) = 1 + V_1(\mu)\tilde{z} + V_2(\mu)\tilde{z}^2 + o(\tilde{z}^2), \tag{2.25}$$

in a neighborhood of the point $z = 1$ where $z = 1 + \tilde{z}$, $\tilde{z} \in \mathbb{C}$. Using the definition of V, we find that

$$\begin{aligned} V_1(\mu) &= \frac{1}{2}\big(y_{12}^1(0,\mu) - y_{21}^1(0,\mu)\big), \\ V_2(\mu) &= \frac{1}{2}\big(y_{12}^2(0,\mu) - y_{21}^2(0,\mu)\big), \quad \mu \in [0,\gamma). \end{aligned} \tag{2.26}$$

It follows from (2.18) that $Y_1(0,\mu) = -Y_0(0,\mu)JD(\mu)$, $\mu \in [0,\gamma)$, where

$$D(\mu) = \int_{-\pi/2}^{0} Y_0^T(s,\mu)H(s,\mu)Y_0(s,\mu)\,ds, \quad \mu \in [0,\gamma).$$

For each value $\mu \in [0,\gamma)$ the matrix $D(\mu) = \|d_{ij}(\mu)\|_1^2$ is symmetric and positive definite; i.e., $d_{11}(\mu) > 0$, $d_{22}(\mu) > 0$, and $d_{12}(\mu) = d_{21}(\mu)$. It follows from (2.22) that

$$Y_0(0,\mu) = \begin{pmatrix} \frac{1}{4}T'(\mu)f(\mu) & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu \in [0,\gamma).$$

Calculating the matrix $Y_1(0,\mu)$, $\mu\in[0,\gamma)$, we find that

$$\frac{\partial V(1,z)}{\partial z}=V_1(\mu)=d_{22}(\mu)\frac{T(\mu)}{4}f(\mu)T'(\mu),\quad \mu\in[0,\gamma). \tag{2.27}$$

It follows from the last formula that condition (2.14) is valid for $\mu=\mu^*$ if and only if $T'(\mu^*)=0$. The first part of the lemma is proven.

Consider a small neighborhood $\{\mu:|\mu-\mu^*|<\delta,\,\mu\in[0,\gamma)\}$ of the critical point $\mu=\mu^*$ and a small neighborhood $\{z:|z-1|<\varepsilon,\,z\in\mathbb{C}\}$ of the point $z=1$. Change the variables $z=1+\tilde z$, $|\tilde z|<\varepsilon$, in (2.7). Using (2.25), transform (2.7) to the form

$$\tilde z(1-2V_1(\mu)-2V_2(\mu))-2V_1(\mu)+o(\tilde z)=0, |\tilde z|<\varepsilon,\quad |\mu-\mu^*|<\delta. \tag{2.28}$$

Show that $V_2(\mu^*)$ is less than zero. From (2.19) we obtain the representation

$$Y_2(0,\mu^*)=Y_0(0,\mu^*)JK,$$

where

$$K=\int\limits_{-\pi/2}^{0}C(s)J\int\limits_{-\pi/2}^{s}C(s_1)ds_1\,ds=\|k_{ij}\|_1^2,$$

$C(s)=Y_0^\top(s,\mu^*)H(s,\mu^*)Y_0(s,\mu^*)=\|c_{ij}(s)\|_1^2$ is a symmetric positive definite matrix for fixed values of the argument $s\in[-\pi/2,0]$. Calculating the matrix

$$Y_2(0,\mu^*)=\begin{pmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{pmatrix},$$

we find $V_2(\mu)=\frac{1}{2}(k_{12}-k_{21})$. Using the definition of K, we obtain

$$V_2(\mu^*)=\frac{1}{2}\int\limits_{-\pi/2}^{0}\left(2c_{12}(s)\int\limits_{-\pi/2}^{0}c_{12}(s)ds-c_{11}(s)\int\limits_{-\pi/2}^{0}c_{22}(s)\,ds-c_{22}(s)\int\limits_{-\pi/2}^{0}c_{11}(s)\,ds\right)ds$$

$$=\frac{1}{2}\left(\int\limits_{-\pi/2}^{0}c_{12}(s)\,ds\right)^2-\int\limits_{-\pi/2}^{0}c_{11}(s)\,ds\int\limits_{-\pi/2}^{0}c_{22}(s)\,ds<0.$$

Consequently,

$$\frac{d^2D(1,\mu^*)}{dz^2}=2(1-2V_2(\mu^*))\neq 0,$$

and the lemma is proven.

Lemma 2.7. *Let f be a thrice continuously differentiable odd function with positive first derivative on an interval $(-\gamma,\gamma)$. As μ increases in a small neighborhood of a critical point μ^*, the root of the characteristic equation goes from the interior to the exterior of the unit disk (from the exterior to the interior) if $T''(\mu^*)>0$ ($T''(\mu^*)<0$).*

Proof. Since $V_2(\mu^*)<0$; therefore, (2.28) has a unique solution in the domain $|\tilde z|<\varepsilon$, $|\mu-\mu^*|<\delta$ [10, p. 26] which is defined by the formula

$$\tilde z=\frac{2\frac{dV_1(\mu^*)}{d\mu}}{1-2V_2(\mu^*)}\bar\mu+o(\bar\mu),\quad \bar\mu=\mu-\mu^*.$$

Using (2.27), we find that

$$\frac{dV_1(\mu^*)}{d\mu}=d'_{22}(\mu^*)\frac{T(\mu^*)}{4}f(\mu^*)T'(\mu^*)+d_{22}(\mu^*)\frac{1}{4}f(\mu^*)(T'(\mu^*))^2$$

$$+d_{22}(\mu^*)\frac{T(\mu^*)}{4}f'(\mu^*)T'(\mu^*)+d_{22}(\mu^*)\frac{T(\mu^*)}{4}f(\mu^*)T''(\mu^*)=d_{22}(\mu^*)\frac{T(\mu^*)}{4}f(\mu^*)T''(\mu^*).$$

Consequently, the directions of the passage of the root of (2.7) through the unit circle are determined by the sign of $T''(\mu^*)$. The lemma is proven.

Theorem 2.1. *Let f be a thrice continuously differentiable odd function with positive first derivative on an interval $(-\gamma, \gamma)$. Suppose that the second derivative of the function T is different from zero at its every critical point. Then for noncritical points $\mu^0 \in (0, \gamma)$ of the function T, the delay differential equation* (2.1) *is stable if $T'(\mu^0) > 0$ and unstable if $T'(\mu^0) < 0$.*

PROOF. In the proof of Assertion 2.1 we demonstrated that (2.1) for $\mu = 0$ has a purely imaginary characteristic exponent $\lambda(0) = i$, while all other characteristic exponents have negative real parts. Using the relation $\rho = e^{\lambda\pi}$ between the characteristic exponents and the eigenvalues of the monodromy operator, we obtain $\rho(0) = -1$. The other eigenvalues of the monodromy operator for $\mu = 0$ lie inside the domain $|\rho| < 1$. Use the relation $\rho = -z^{-2}$ between the eigenvalues ρ of the monodromy operator and the roots z of (2.7). Then the eigenvalue $\rho(0) = -1$ is associated with the pair of roots $z_{1,2}(0) = \pm 1$ and the eigenvalues ρ inside the domain $|\rho| < 1$, with roots in the domain $|z| > 1$. By Lemma 2.3, there is an eigenvalue $\rho_1(\mu) = -1$, $\mu \in [0, \gamma)$, associated with the pair of roots $z_{1,2}(\mu) = \pm 1$, $\mu \in [0, \gamma)$, of (2.7).

Observe that the point $\mu = 0$ is critical and the signs of the values of T'' alternate at successive critical points.

Let $T''(0) > 0$. Consequently, $T'(\mu) > 0$ for $0 < \mu < \mu_1$, where μ_1 is the critical point nearest to $\mu = 0$. Then, by Assertion 2.1, equation (2.1) is stable for small positive μ. The multiple roots $z_{1,2} = \pm 1$ split and two roots go to the domain $|z| > 1$. By Lemma 2.6, no roots of (2.7) may appear in the domain $|z| < 1$ with the increase of μ on the interval $0 < \mu < \mu_1$. Consequently, we have stability. At the critical point μ_1 we have $T''(\mu_1) < 0$ and $T'(\mu) < 0$ for $\mu_1 < \mu < \mu_2$, where μ_2 is the critical point nearest to μ_1 from the right. By Lemma 2.7, as the parameter μ passes through the point μ_1, a pair of roots of (2.7) enters the domain $|z| < 1$ and stability changes for instability. On the interval $\mu_1 < \mu < \mu_2$, the number of roots in the domain $|z| < 1$ cannot change and hence we have instability on this interval.

Let $T''(0) < 0$. Consequently, $T'(\mu) < 0$ for $0 < \mu < \mu_1$, where μ_1 is the critical point nearest to $\mu = 0$. Then, by Assertion 2.1, equation (2.1) is unstable for small positive μ. The multiple roots $z_{1,2} = \pm 1$ split, and two roots go to the domain $|z| < 1$. By Lemma 2.6, with the increase of μ on the interval $0 < \mu < \mu_1$, the number of roots of (2.7) in the domain $|z| < 1$ cannot change. Consequently, we have instability on this interval. At the critical point μ_1 we have $T''(\mu_1) > 0$ and $T'(\mu) > 0$ for $\mu_1 < \mu < \mu_2$, where μ_2 is the critical point nearest to μ_1 from the right. By Lemma 2.7, upon the passage through the point μ_1 a pair of roots of (2.7) exits from the domain $|z| < 1$ and instability changes for stability. On the interval $\mu_1 < \mu < \mu_2$ the number of roots in the domain $|z| < 1$ cannot change and hence we have stability on this interval.

This analysis demonstrates that stability of (2.1) for noncritical points of the function T is determined by the sign of the first derivative of the period function T at these points. The theorem is proven.

3. Stability of Periodic Solutions to a Nonlinear Delay Differential Equation

We return to the problem of stability of an antisymmetric periodic solution to a delay differential equation.

Theorem 3.1. *Let f be a thrice continuously differentiable odd function with positive first derivative on an interval $(-\gamma, \gamma)$. Suppose that the second derivative of T is different from zero at its every critical point and a noncritical point $\mu^0 \in (0, \gamma)$ of the function T is a root of the equation $T(\mu) = 4\tau$. Then the antisymmetric periodic solution to the delay differential equation* (1.1) *corresponding to this root is stable* (*unstable*) *if $T'(\mu^0) > 0$ ($T'(\mu^0) < 0$).*

PROOF. Suppose that the condition $T'(\mu^0) < 0$ is satisfied for the noncritical point $\mu^0 \in (0, \gamma)$. It follows from the proof of Theorem 2.1 that there is a pair of roots of (2.7) in the domain $|z| < 1$. This pair is associated with an eigenvalue ρ^0 of the monodromy operator whose absolute value is greater than one. Then, by the theorem on instability in linear approximation [11, p. 287], a periodic solution to the delay differential equation (1.1) is unstable.

Suppose that the condition $T'(\mu^0) > 0$ is satisfied at a noncritical point $\mu^0 \in (0, \gamma)$. It follows from the proof of Theorem 2.1 that all roots of (2.7) lie in the domain $|z| > 1$ except for the pair $z^0_{1,2} = \pm 1$.

This pair $z^0_{1,2}$ is associated with the eigenvalue $\rho^0 = -1$ of the monodromy operator. The roots of (2.7) in the domain $|z| > 1$ are associated with the eigenvalues of the monodromy operator in the domain $|\rho| < 1$.

Show that the eigenvalue $\rho^0 = -1$ is simple. Using the relation $z = i\rho^{-\frac{1}{2}}$ between the roots of (2.7) and the eigenvalues of the monodromy operator, write down the characteristic equation for finding ρ:

$$\widetilde{D}(\rho, \mu^0) = -\rho^{-1} + 2i\rho^{-\frac{1}{2}} V(-i\rho^{-\frac{1}{2}}, \mu^0) + 1 = 0.$$

Then

$$\frac{\widetilde{D}(\rho, \mu^0)}{d\rho}\bigg|_{\rho=-1} = 1 - V(1, \mu^0) - \frac{V(1, \mu^0)}{dz} = -\frac{V(1, \mu^0)}{dz} \neq 0.$$

Using an analog of the Andronov–Vit theorem for functional-differential equations [11, p. 287], we conclude that a periodic solution to the delay differential equation (1.1) is stable.

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